Math 246B Lecture 10 Notes

Daniel Raban

January 31, 2019

1 Relationships Between Compactly Supported and Holomoprhic Functions

1.1 Solving the inhomogeneous Cauchy-Riemann equation

Last time, we proved the Cauchy integral formula for non-holomorphic functions.

Definition 1.1. When $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{C}$ is a function, we define the **support** of f $\operatorname{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$ (closure with respect to Ω).

Definition 1.2. When $0 \leq k \in \mathbb{N} \cup \{\infty\}$, let $C_0^k(\Omega) = \{u \in C^k(\Omega) : \operatorname{supp}(u) \subseteq \Omega \text{ is compact}\}$.

Proposition 1.1. *Let $\psi \in C_0^k(\mathbb{C})$. Then there exists $u \in C^k(\mathbb{C})$ solving the inhomogeneous Cauchy-Riemann equation*

$$\frac{\partial u}{\partial \overline{z}} = \psi.$$

Proof. Apply Cauchy's integral formula.

$$\psi(z) = -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} \, L(d\zeta)$$

Make the substitution $\zeta \mapsto \zeta + z$.

$$\begin{aligned} &= -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta + z) \frac{1}{\zeta} L(d\zeta) \\ &= \frac{\partial \psi}{\partial \overline{\zeta}} \left(-\frac{1}{\pi} \iint \frac{\psi(\zeta + z)}{\zeta} L(d\zeta) \right). \end{aligned}$$

We can differentiate under the integral sign because $1/\zeta \in L^1_{\text{loc}}$, and $\psi \in C_0^1$. So we can take

$$u(z) = -\frac{1}{\pi} \iint \frac{\psi(\zeta)}{\zeta - z} \, L(d\zeta) \overset{\zeta \to \zeta + z}{=} \iint \frac{\psi(\zeta - z)}{\zeta} L(d\zeta) \in C^k(\mathbb{C}). \qquad \square$$

1.2 Bounds on derivatives of holomorphic functons

Proposition 1.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $K \subseteq \Omega$ be compact. Then there exists $\psi \in C_0^1(\Omega)$ such that $\psi = 1$ in a neighborhood of K.*

Here, ψ is called a **cutoff function**.

Proof. Let $\delta > 0$ be such that $\operatorname{dist}(x, K) \geq \delta$ for any $z \in \mathbb{C} \setminus \Omega$, and let $\tilde{K} = \{z \in \mathbb{C} : \operatorname{dist}(z, K) < \delta/2\}$. $\tilde{K} \subseteq \Omega$ is compact. Let also $\varphi \in C^1(\mathbb{C})$ with $\varphi \geq 0$, $\varphi(z) = 0$ for $|z| \geq 1$, and $\iint \varphi = 1$. For example, we can take

$$\varphi(z) = \begin{cases} B(1 - |z|^2)^2 & |z| \leq 1 \\ 0 & |z| > 1 \end{cases}$$

for some B chosen so that $\iint \varphi = 1$. Let $\varphi_t(z) = t^{-2}\varphi(z/t)$, where $t > 0$. Then $\operatorname{supp}(\varphi_t) \subseteq \{|z| \leq t\}$, and $\iint \varphi_t = 1$ for any t.

Now consider

$$\psi(z) = \mathbb{1}_{\tilde{K}} * \varphi_{\delta/3} = \iint \varphi_{\delta/3}(z - \zeta)\mathbb{1}_{\tilde{K}}(\zeta)\, L(d\zeta).$$

Then $\psi \in C^1(\mathbb{C})$. If $\psi(z) \neq 0$, then there exists $\zeta \in \tilde{K}$ such that $|z - \zeta| \leq \delta/3$. We get that

$$\operatorname{dist}(z, K) \leq \operatorname{dist}(\zeta, K) + |z_\zeta| \leq \frac{\delta}{2} + \frac{\delta}{3} \leq \frac{5}{6}\delta < \delta.$$

So $\operatorname{supp}(\psi)$ is a compact subset of Ω. That is, $\psi \in C_0^1(\Omega)$. Moreover, for z with $\operatorname{dist}(z, K) \leq \delta/12$, $\operatorname{dist}(z - z\zeta, K) \leq \operatorname{dist}(z, K) + |\zeta| < \delta/2$, so

$$\psi(z) - 1 = \iint (\mathbb{1}_{\tilde{K}}(\zeta) - 1)\varphi_{\delta/3}(z - \zeta)\, L(d\zeta) = \iint (\mathbb{1}_{\tilde{K}}(z - \zeta) - 1)\varphi_{\delta/3}(\zeta) L(d\zeta) = 1. \quad \square$$

Remark 1.1. This construction is valid in any Euclidean space, not just $\mathbb{C}$.

Proposition 1.3. *Let $f \in \operatorname{Hol}(\Omega)$. For any compact $K \subseteq \Omega$ and any open neighborhood $\omega \subseteq \Omega$ of K, we have for $j = 0, 1, 2, \dots$ that there exists a constant $C_j = C_{j,\omega,K}$ such that*

$$\sup_{z \in K} |f^{(j)}(z)| \leq C_j \|f\|_{L^1(\omega)}.$$

Proof. Let ψ be as in the previous proposition. Apply Cauchy's integral formula to the function $\psi f \in C_0^1(\Omega) \subseteq C_0^1(\mathbb{C})$:

$$(\psi f)(z) = -\frac{1}{\pi} \iint \underbrace{\frac{\partial}{\partial \bar{\zeta}}(\psi f)(\zeta)}_{= \frac{\partial \psi}{\partial \bar{\zeta}} f} \frac{1}{\zeta - z}\, L(\zeta)$$

for all $z \in \mathbb{C}$. So for z in a neighborhood of K,

$$f(z) = -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{f(\zeta)}{\zeta - z} L(d\zeta).$$

where the region of integration is $\operatorname{supp}(\frac{\partial \psi}{\partial \bar{\zeta}}) \cap K$. Differentiating under the integral sign, we get

$$f^{(j)}(z) = -\frac{j!}{\pi} \iint \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{f(\zeta)}{(\zeta - z)^{j+1}} L(d\zeta).$$

So

$$\|f^{(j)}\|_{L^\infty(K)} \leq \frac{j!}{\pi \delta^{j+1}} \left\| \frac{\partial \psi}{\partial \bar{\zeta}} \right\|_{L^\infty} \|f\|_{L^1(\omega)},$$

where $|\zeta - z| \geq \delta$. $\square$